Minimal Root Sensitivity in Linear Systems

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Nominal performance, root sensitivity, and stability are important control design considerations. This paper deals only with the first two of these concerns. A lower bound is derived for root sensitivity and necessary and sufficient conditions are given to achieve this minimum. This is the main result of the paper. In addition, an optimal output feedback control problem is discussed which penalizes an index related to root sensitivity. Two hazards are clarified concerning root sensitivity designs. First, we illustrate that root sensitivity has nothing to do with stability. Second, we show that normality of the plant matrix is the necessary and sufficient condition for minimal root sensitivity, but that normality measures can be convex even when root sensitivity measures are not. Hence, the optimization problems are quite different, and normality objectives cannot always be substituted for root sensitivity objectives.

Introduction

THE modal data for physical systems is rarely well known. This can make performance predictions unreliable in both open-loop and feedback control problems. This paper documents the smallest possible sensitivity of eigenvalues λ_i with respect to the independent plant parameters in linear systems of the form

$$\dot{x}=Ax, x\in\mathcal{R}^n \tag{1}$$

That is, the norm of the root sensitivity matrix

$$\frac{\partial\lambda_i}{\partial A}=\begin{bmatrix}\frac{\partial\lambda_i}{\partial A_{11}} & \cdots & \frac{\partial\lambda_i}{\partial A_{1n}}\\ \vdots & & \vdots\\ \frac{\partial\lambda_i}{\partial A_{n1}} & \cdots & \frac{\partial\lambda_i}{\partial A_{nn}}\end{bmatrix} \tag{2}$$

and the lower bound of its norm are of interest. Second, a metric related to root sensitivity is added to the optimal output feedback problem in an attempt to achieve a compromise between performance and root sensitivity.

This approach to system control design is illustrated by a simple example to point out certain pitfalls in sensitivity analysis and design which have not been clarified previously. The first pitfall is the fact that sensitivity has nothing to do with "stability" and yet these two objectives are often substituted in the literature. To understand the second pitfall, we note that the necessary and sufficient condition for minimal root sensitivity is the normality of the plant matrix. This fact is used by researchers to suggest that normality is a good design goal,[1] but in fact the normality measure can be a tractable convex function even when root sensitivity is not. The second pitfall, then, is that even if small root sensitivity were a desirable design goal, the substitution of normality measures for root sensitivity measures is not always appropriate, as our example will illustrate.

The norm squared of a matrix should be denoted by

$$\|[\cdot]\|^2=\mathrm{tr}[\cdot]^*[\cdot], \qquad \mathrm{tr}[\cdot]\triangleq\mathrm{trace}[\cdot] \tag{3a}$$

and the norm squared of a vector shall be denoted by

$$\|(\cdot)\|^2=(\cdot)^*(\cdot) \tag{3b}$$

where the asterisk denotes complex conjugate transpose. Results herein are limited to the case of distinct eigenvalues for A.

Construction of a Root Sensitivity Metric

The sensitivity of the ith eigenvalue $\partial\lambda_i/\partial A$ is an $n\times n$ matrix denoted by $S_i\triangleq\partial\lambda_i/\partial A$. The norm of S_i from Eqs. (3) is

$$\|S_i\|^2\triangleq\mathrm{tr}S_i^*S_i=\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\left[\left(\frac{\partial\lambda_i}{\partial A_{\alpha\beta}}\right)^2\right] \tag{4}$$

The complete root sensitivity metric of interest is

$$s\triangleq\sum_{i=1}^{n}\|S_i\|^2\triangleq\sum_{i=1}^{n}\left\|\frac{\partial\lambda_i}{\partial A}\right\|^2 \tag{5}$$

Thus, from the point of view of root sensitivity, a system design with a large value of s might be considered less desirable than a system design with a small value of s. This would be true if the analyst is specifically concerned that root locations remain fixed in the presence of parameter uncertainties.

In order to compute the root sensitivity metric, we will assume that A has a linearly independent set of eigenvectors e_i

$$Ae_i=e_i\lambda_i \qquad i=1,\ldots,n \tag{6}$$

The reciprocal basis vectors ℓ_i are defined by

$$E\triangleq[e_1,\ldots,e_n], \quad \begin{bmatrix}\ell_1^*\\ \vdots\\ \ell_n^*\end{bmatrix}\triangleq E^{-1}$$

hence

$$\ell_i^*e_j=\delta_{ij} \tag{7}$$

Multiplying Eq. (6) from the left by ℓ_i^* using Eq. (7), yields the eigenvalues in terms of A its eigenvectors, and its reciprocal basis vectors,

$$\lambda_i=\ell_i^*Ae_i \tag{8}$$

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Differentiation of the scalar equation (8) with respect to A provides the required sensitivity $\partial\lambda_i/\partial A$. To derive this result two identities from linear algebra are required

$$\mathrm{tr}AB = \mathrm{tr}BA \tag{9}$$

$$\frac{\partial}{\partial A}(\mathrm{tr}AB) = \frac{\partial}{\partial A}(\mathrm{tr}BA) = B^T \tag{10}$$

where Eqs. (9) and (10) hold for real or complex matrices B and A, and Eq. (10) holds if the elements of A are independent. Hence from Eq. (8), using Eq. (9) and (10),

$$\frac{\partial\lambda_i}{\partial A} = \frac{\partial}{\partial A}[\mathrm{tr}A(e_i\ell_i^*)] = (e_i\ell_i^*)^T = \bar{\ell}_i e_i^T \tag{11}$$

where the overbar denotes complex conjugate.

Equation (11) can also be obtained from Jacobi's formula[2] for small root perturbations

$$\Delta\lambda_i = \ell_i^*\Delta A e_i \tag{12}$$

where $\Delta\lambda_i$ is the change (to first-order approximation) of λ_i in the presence of a perturbation of A to $A+\Delta A$. Equation (11) is also a more compact generalization of the eigenvalue sensitivity used in Ref. 3. The norm [Eq. (4)] now may be written

$$\|S_i\|^2 \triangleq \mathrm{tr}S_i^*S_i = \mathrm{tr}(\bar{\ell}_i e_i^T)^*(\bar{\ell}_i e_i^T) = \|\ell_i\|^2\|e_i\|^2 \tag{13}$$

where the last equality requires use of identity equation (9) again, and where

$$\|\ell_i\|^2 = \ell_i^*\ell_i,\ \|e_i\|^2 = e_i^*e_i \tag{14}$$

One may note the similarity between Eq. (11) and the weak differential of λ, given by Eq. (6.2.3) of Ref. 4. Also note the similarity between Eq. (13) and the upper bound of the weak derivative of λ_i provided in Ref. 4 (top of p. 235). This paper seeks the lower bounds of root sensitivity rather than the upper bounds. Otherwise, the nature of the results are similar (see theorem 6.2.4 of Ref. 4). Also note that the sensitivity model [Eqs. (11) and (13)] is different from the one used by Wilkinson.[5] Wilkinson, and later on, others[6] use the inner product of normalized left and right eigenvectors: $s_i = r_i^* e_i$, with $\|r_i\| = 1 = \|e_i\|$. Our result [Eq. (11)] involves the outer product of the reciprocal basis vector and the (right) eigenvector without requiring any special normalization. Both points of view use left eigenvectors, but with different normalizations: left eigenvectors normalized to unit length, as opposed to that choice of left eigenvectors corresponding to a reciprocal basis of the (right) eigenvector. We choose the latter. [Note, however, that if we used Wilkinson's model with normalized eigenvectors and normalized reciprocal basis vectors, his sensitivity index would be merely the reciprocal of Eq. (13)]. Since reciprocal basis vectors are uniquely determined from the eigenvectors, only *one* normalization is required (the initial one on the eigenvectors) in our metric, as opposed to two normalizations required by Wilkinson[5] and Postlethwaite.[6] We also would like the results to be independent of the particular normalization chosen for the eigenvectors; our approach has this property.

Gilbert[7] has computed the sufficient conditions for minimality for a variety of sensitivity indices:

$$\left\|\frac{\partial|\lambda_i|}{\partial A}\right\|,\ \left\|\frac{\partial \mathrm{Re}\lambda_i}{\partial A}\right\|,\ \text{and}\ \left\|\frac{\partial \mathrm{Im}\lambda_i}{\partial A}\right\|$$

In addition, he has proved the differentiability which we merely assumed in Eqs. (2) and (11). We seek necessary and sufficient conditions for minimality of Eq. (13).

Now, consider Eq. (13) again. The Schwartz inequality[4] holds for any two vectors, ℓ_i, e_i

$$|\ell_i^* e_i| \le \|\ell_i\|\ \|e_i\| \tag{15}$$

Since the particular vectors ℓ_i, e_i are related by Eq. (8),

$$\ell_i^* e_i = 1 \tag{16}$$

Equations (15) and (16) lead immediately to

$$\|\ell_i\|\ \|e_i\| \ge 1 \tag{17}$$

Squaring both sides of Eq. (17), and using Eq. (13), leads to

$$\|S_i\|^2 \ge 1 \tag{18}$$

The equality in Eq. (17), and hence in Eq. (18), holds if, and only if, ℓ_i and e_i are colinear ($\ell_i = e_i$) (Ref. 4). From linear algebra[8], $\ell_i = e_i$ if, and only if, A is normal ($AA^* = A^*A$). Thus the main theoretical results of the paper are summarized as follows.

Theorem 1: Let (λ_i, e_i, ℓ) be the ith eigenvalue, eigenvector and its reciprocal basis vector associated with the real matrix A. If A has a linearly independent set of eigenvectors e_i, $i = 1,\ldots,n$ then

$$\left\|\frac{\partial\lambda_i}{\partial_A}\right\| \ge 1 \qquad i = 1,\ldots,n \tag{19a}$$

where the lower bound

$$\left\|\frac{\partial\lambda_i}{\partial A}\right\| = 1 \qquad i = 1,\ldots,n \tag{19b}$$

is achieved if and only if $AA^* = A^*A$. The sensitivity metric [Eq. (5)] is bounded from below by

$$s \ge n \tag{20}$$

and the minimum sensitivity $s = n$ is achieved if and only if A is normal ($AA^* = A^*A$).

The theorem provides necessary and sufficient conditions for minimum root sensitivity. If one wishes to keep roots relatively fixed in the presence of parameter variations, theorem 1 indicates that normality of A is a necessary and sufficient condition for a globally minimal value of $\|\partial\lambda_i/\partial A\|$. Gilbert[7] has shown that symmetry of A is a sufficient condition for a global minimum of $|\partial \mathrm{Re}\lambda_i/\partial A|$. Since a symmetric A is also normal, and has only real eigenvalues, this is in agreement with theorem 1. The next section suggests a means to incorporate this information into the output feedback control design problem.

Output Feedback Design

Parameter sensitivity has long been a concern in optimal control. Some authors[9,10] have suggested modifying a quadratic performance index by the addition of trajectory sensitivity terms

$$\sum_i \left\|\frac{\partial y}{\partial p_i}\right\|^2$$

(where p_i, $i = 1,\ldots,r$ represent the uncertain parameters). The resulting computational burdens are very great indeed, since the dimension of the constraint (state) equation becomes $n(1+r)$. Also, mimimizing output sensitivity does not necessarily keep root sensitivity small. Previously we showed that root sensitivity is minimized when A is normal. It has also been shown[11] that the robustness bound for a certain class of parameter errors is maximized when the plant matrix A is normal.

Normality has often been used as a design goal. MacFarlane and Hung[1] recently stated, "...an approximation to normality is something which one strives to achieve in the feedback design process. Thus, normal systems and their properties play a key role in the formulation and implementation of what we call the quasi-classical approach to feedback systems." The suggestion here, of course, is that normal matrices A are desirable, and that approximations of normal A are desirable. Therefore, with this background, together with the motivation of theorem 1, we shall design a controller to yield a "more normal" closed-loop plant matrix, rather than attempting to minimize root sensitivity directly. We shall return later, by way of example, to the question of how normality relates to root sensitivity outside the neighborhood of the global minimum of root sensitivity. This comparison should provide considerable insight toward the understanding of "normal" design objectives, such as in Refs. 1 and 6.

Consider a new performance index for optimization that includes an "abnormality" penalty

$$\mathcal{V} = \lim_{t\to\infty} \mathcal{E}(\|y\|_Q^2 + \|u\|_R^2) + \beta\|(A+BGM)(A+BGM)^T - (A+BGM)^T(A+GBM)\|_{C^TQC}^2 \tag{21}$$

subject to the state equations

$$\dot{x} = Ax + Bu + Dw$$

$$y = Cx$$

$$z = Mx + v$$

$$u = Gz$$

$$\mathcal{E}\binom{w}{v} = 0$$

$$\mathcal{E}\binom{w(t)}{v(t)}(w^T(\tau), v^T(\tau)) = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}\delta(t-\tau)$$

$$\mathcal{E}\binom{w(t)}{v(t)}x^T(0) = 0, \qquad t>0 \tag{22}$$

When β is much smaller than the norms of Q and R the solution tends toward the standard optimal control result.[12,13] On the other extreme, when β is chosen much larger than the norms of Q and R, the closed-loop system approaches the smallest possible root sensitivity. [From theorem 1 note that root sensitivity is minimized if and only if $A+BGM$ is normal, in which case the latter term in Eq. (21) is zero.]. Other choices of weights on the matrix norm may be chosen besides C^TQC. This choice is suggested only to make the sensitivity weight C^TQC the same as the state weight in $y^TQy = x^T[C^TQC]x$. That is, if a certain state is not important enough to penalize in the cost function, its sensitivity is probably not important either.

Using the same matrix norm as in previous sections, and defining $\mathcal{A} \triangleq A+BGM$, $\mathcal{V}$ becomes

$$\mathcal{V} = \mathrm{tr}\,P(C^TQC + M^TG^TRGM) + \mathrm{tr}\,VG^TRG + \beta\,\mathrm{tr}[(\mathcal{A}\mathcal{A}^T - \mathcal{A}^T\mathcal{A})]C^TQC[(\mathcal{A}\mathcal{A}^T - \mathcal{A}^T\mathcal{A})] \tag{23}$$

when P is the solution of

$$0 = P(A+BGM)^T + (A+BGM)P + BGVG^TB^T + DWD^T \tag{24}$$

The necessary conditions for the optimum G are obtained by augmenting constraint equation (24) to Eq. (23) via Lagrange multiplier matrix K and differentiating the augmented $\mathcal{V}$ with respect to P, K, and G. The result is that K and G must satisfy

$$0 = K(A+BGM) + (A+BGM)^TK + M^TG^TRGM + C^TQC \tag{25}$$

$$0 = RGMPM^T + RGV + B^TKPM^T + B^TKBGV + \Psi \tag{26}$$

where

$$\Psi \triangleq \beta B^T\{[(\mathcal{A}\mathcal{A}^T - \mathcal{A}^T\mathcal{A})C^TQC + C^TQC(\mathcal{A}\mathcal{A}^T - \mathcal{A}^T\mathcal{A})]\mathcal{A} - \mathcal{A}[(\mathcal{A}\mathcal{A}^T - \mathcal{A}^T\mathcal{A})C^TQC + C^TQC(\mathcal{A}\mathcal{A}^T - \mathcal{A}^T\mathcal{A})]\}M^T \tag{27}$$

These results are summarized as follows.

Theorem 2: The necessary conditions for minimizing Eq. (21) subject to the constraints (22) are given by Eqs. (24-26) and (27).

It has been shown[12,13] that the necessary conditions for the output feedback solution for system (22) to minimize the standard quadratic cost function [Eq. (21) with $\beta = 0$] are given by Eqs. (24-26) with $\Psi = 0$. Various suboptimal strategies for approximating the solution of Eqs. (24-26) (with $\Psi = 0$) may be found in the literature.[13]

The following conclusion is a very special case (and not usually practical) of the sensitivity minimization problem but the results are easily given. For an arbitrary A, the matrix $A+BGM$ can be made normal (by choice of G) only if rank B = rank $M = n$. Hence, we have the following.

Theorem 3: The minimum sensitivity $s = n$ can be *guaranteed* by output feedback control for an *arbitrary* A if and only if rank B = rank $M = n$. Furthermore, the control gain in this case is not unique. Two gains that provide minimum sensitivity are

$$G = -B^{-1}AM^{-1} \tag{28}$$

$$G = B^{-1}A^TM^{-1} \tag{29}$$

Proof: Substitute Eqs. (28) and (29) into the normality condition for minimum sensitivity

$$(A+BGM,)(A+BGM)^T - (A+BGM)^T(A+BGM) = 0 \tag{30}$$

to see that condition (30) holds.

Application of Closed-Loop Root Sensitivity Design

Example 1

The pitch motion of a rigid aircraft is governed by[14]

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -1/\tau & 1 \\ -\omega_o^2 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_o^2\mathcal{Q} \end{bmatrix}(u+w) \tag{31}$$

where α is the angle of attack, q the pitch rate, u the elevator angle, τ the lifting time constant, ω_0 the undamped pitch natural frequency, and $\mathcal{Q}$ the elevator effectiveness. In the open-loop case, $(u+w) = 0$, we are interested in the root sensitivity properties of Eq. (31).

Note from theorem 1 that minimum sensitivity

$$s = \sum_{i=1}^{2}\left\|\frac{\partial\lambda_i}{\partial A}\right\|^2 = 2$$

is achieved if and only if A is normal. Computing the "abnormality" matrix $(AA^* - A^*A)$ yields

$$AA^* - A^*A = \begin{bmatrix} 1-\omega_o^4 & \frac{1}{\tau}(1+\omega_o^2) \\ \frac{1}{\tau}(1+\omega_o^2) & \omega_o^4 - 1 \end{bmatrix} \tag{32}$$

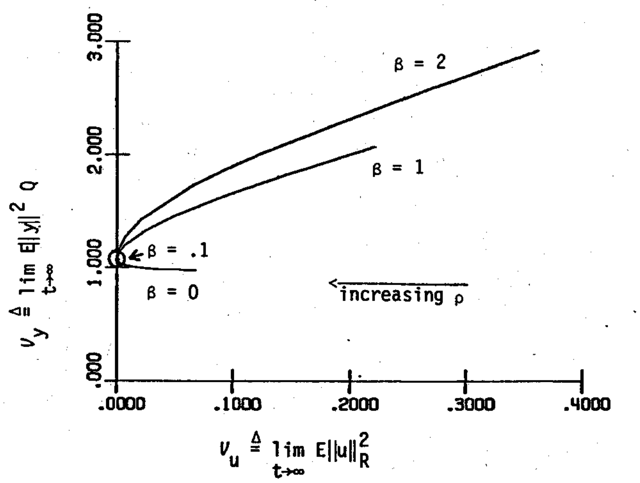

Fig. 1 Output cost $\mathcal{V}_y$ vs input cost $\mathcal{V}_u$ [$\tau=0.7$, $\omega_o=1.0$].

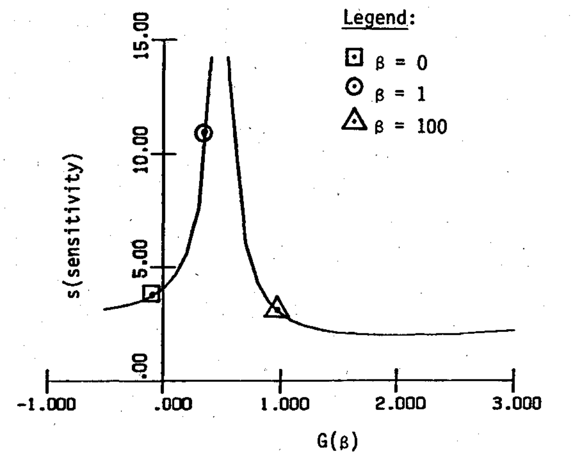

Fig. 3 Root sensitivity s vs feedback gain $G(\beta)$ [$\tau=0.7$, $\omega_o=1.0$, $\rho=1$].

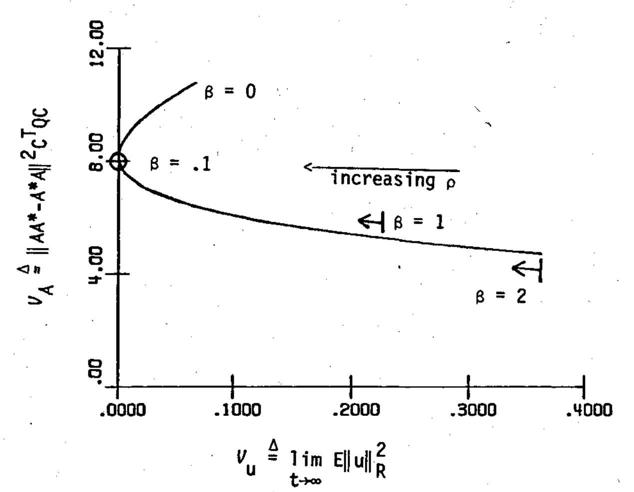

Fig. 2 Abnormality cost $\mathcal{V}_{\alpha}$ vs input cost $\mathcal{V}_u$ [$\tau=0.7$, $\omega_o=1.0$].

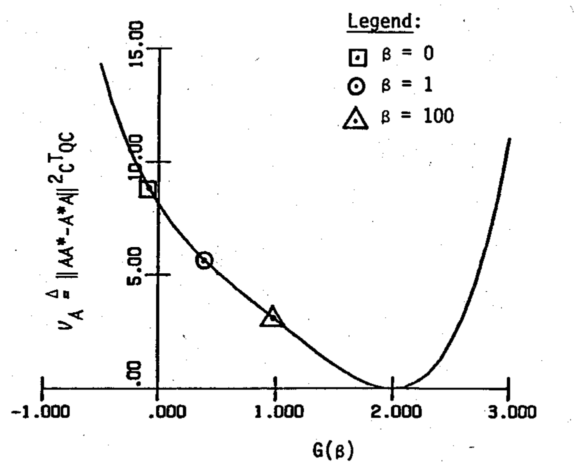

Fig. 4 Abnormality cost $\mathcal{V}_{\alpha}$ vs feedback gain $G(\beta)$ [$\tau=0.7$, $\omega_o=1.0$, $\rho=1$].

Thus, root sensitivity takes on its absolute minimum when $\omega_0=1$, $1/\tau=0$. This, of course, is not a practical possibility for the aircraft. Now consider the output feedback design of the previous section.

Example 2

For the aircraft in example 1, let the angle-of-attack measurement be made $z=\alpha+v$, where $\mathcal{E}[v]=0$, $\mathcal{E}[v(t)v(\tau)]=\delta(t-\tau)$ describes the white measurement noise v and $\mathcal{E}[w]=0$, $\mathcal{E}[w(t)w(\tau)]=\delta(t-\tau)$ describes the white actuator noise w. For the system (31) design a measurement feedback control law for regulating u such that

$$\mathcal{V}=\lim_{t\to\infty}\{\mathcal{E}(q^2+\rho u^2)+\beta\|(A+BGM)(A+BGM)^T - (A+BGM)^T(A+BGM)\|^2_{C^TC}\} \tag{33}$$

is minimized. [We have assumed $Q=1$, $R=\rho$, and $C=[0,1]$; with these values, Eq. (21) yields Eq. (33).]

The solution is provided by Eq. (26), where P is obtained from Eq. (24),

$$P=\begin{bmatrix} \tau & 1 \\ 1 & \dfrac{1}{\tau}+\omega_o^2\tau(1-G) \end{bmatrix}\frac{\omega_o^2(G^2+1)}{2(1-G)} \tag{34}$$

and K is obtained from

$$K=\begin{bmatrix} \frac{1}{2}[G^2\rho\tau-\omega_o^2\tau(G-1)] & -\frac{1}{2} \\ -\frac{1}{2} & \dfrac{\rho G^2\tau^2+1-\omega_o^2\tau^2(G-1)}{2\tau\omega_o^2(1-G)} \end{bmatrix} \tag{35}$$

For the aircraft example, Eq. (26) yields a fifth-order equation in G as a function of β, ρ, ω_o^2, and τ. For given values of β, ρ, ω^2 and τ this equation will yield five candidate values of G. The optimal G is that root that minimizes the cost function. In the present case $G<1$ is required for stability. If none of the real candidate values of G are <1, then analysis of this problem shows that a decrease in β will drive G toward more stable values. This means that normality and stability are diametrically *opposed* in this circumstance.

Assuming $\omega^2=1$ and $\tau=0.7$ (this corresponds to damping ratio $=0.71$). Setting $\beta=0$ and $\rho=1$ yields the standard optimal measurement feedback control

$$G=-0.118 \tag{36}$$

and setting $\beta=\infty$, $\rho\ll\infty$ yields the optimally sensitive design

$$G=2.0 \tag{37}$$

This choice of G in Eq. (37) forces the closed-loop system matrix to be symmetric (hence normal)

$$A+BGM=\begin{bmatrix} -1/\tau & 1 \\ -\omega_o^2+\omega_o^2 G & 0 \end{bmatrix}=\begin{bmatrix} -1/\tau & 1 \\ I & 0 \end{bmatrix} \tag{38}$$

and by theorem 1, the sensitivity is at its minimum in this case. Note, however, that stability is lost by this minimum sensitivity design, ($G<1$ is required for stability). Thus, minimally sensitive designs might not be stable.

Figure 1 shows the tradeoff between output performance

$$\mathcal{V}_y \triangleq \lim_{t\to\infty} \mathcal{E}\|y\|_Q^2$$

and the control effort

$$\mathcal{V}_u \triangleq \lim_{t\to\infty} \|u\|_R^2$$

Figure 2 shows the tradeoff between abnormality of $\mathcal{Q}=(A+BGM)$, $\mathcal{V}_{\mathcal{Q}}=\|\mathcal{Q}\mathcal{Q}^*-\mathcal{Q}^*\mathcal{Q}\|_{C^TQC}$, and the control effort. For both figures, $\omega_0^2=1$, $\tau=0.7$, β and ρ vary. In the standard output feedback design ($\beta=0$), the output performance is improved with an increase in control effort (Fig. 1) whereas the abnormality index greatly increases with control effort (Fig. 2 with $\beta=0$). Note also that $\beta>1$ is not desired, since larger values of β do *not* yield substantially larger abnormality reductions (Fig. 2) but *do* accelerate the degradation of the nominal output performance (Fig. 1).

Now, if we were to assume that abnormality was a consistent metric for root sensitivity (i.e., if both were convex functions), then we could use Figs. 1 and 2 as design tools to choose the feedback law that achieves the best compromise between performance and root sensitivity (represented by abnormality). However, abnormality is *not* a consistent index of root sensitivity, except in the local neighborhood of the globally optimal (normal) condition, as illustrated by Figs. 3 and 4. From these figures it is obvious that (for this example) abnormality and root sensitivity are only consistent indices *above* $G=0.49$ ($\beta=1$) and are actually *opposing* indices below that point. From Figs. 1 and 2, we stated that we would not choose $\beta>1$ if designing for performance/normality. Hence, for this problem, using Figs. 1 and 2 would place us in the realm where abnormality is an *inconsistent* index for root sensitivity. Note also that the sensitivity $\to\infty$ when we have repeated roots ($G=0.49$).

Conclusions

An explicit expression for a scalar metric of root sensitivity is given in terms of the left and right eigenvectors of the system, so that sensitivity of each eigenvalue with respect to the plant matrix may be readily computed. A necessary and sufficient condition for minimum root sensitivity is that the plant matrix of the state equations be normal.

An "abnormality" term therefore may be added to the traditional quadratic performance matrix of the optimal control in an attempt to force the resulting closed-loop plant to be "nearly normal." The necessary conditions are given for the solution of this problem and an example gives some practical insights. The solution to this problem points out that while "normality" implies minimal root sensitivity, minimizing abnormality to get "near normality" will not decrease root sensitivity consistently. In fact, decreasing abnormality may very well *increase* root sensitivity, especially when two roots are close together.

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